Recursive Subtyping for All

May 26 & 28, 2025

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Outline

Recursive Types: What and Why?

- Useful for modeling structures that refer to themselves:
  - Algebraic Data Types (e.g., Lists, Trees)
  - Recursive Objects

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As a Recursive Type

Can be represented as:

$$\mu\alpha.\{\mathsf{x} : \mathsf{Int}, \mathsf{y} : \mathsf{Int}, \mathsf{move} : \mathsf{Int} \to \mathsf{Int} \to \alpha\}$$

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Two Main Approaches

- **Equi-recursive types**: $\mu\alpha.\, A$ and its unfolding are considered equal.
- **Iso-recursive types**: $\mu\alpha.\, A$ and its unfolding $A[\mu\alpha.\, A/\alpha]$ are distinct but isomorphic up to unfold and fold operations. (Focus of this talk)

Iso-Recursive Types: Explicit Folding/Unfolding

- A recursive type $\mu\alpha.\, A$ is **not equal** to its one-step unfolding $A[\mu\alpha.\, A/\alpha]$.
- They are only isomorphic, requiring explicit conversion operators:
  - $\mathsf{unfold}[\mu\alpha.\, A] : \mu\alpha.\, A \to A[\mu\alpha.\, A/\alpha]$
  - $\mathsf{fold}[\mu\alpha.\, A] : A[\mu\alpha.\, A/\alpha] \to \mu\alpha.\, A$

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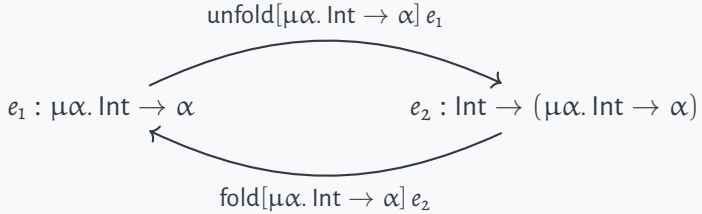

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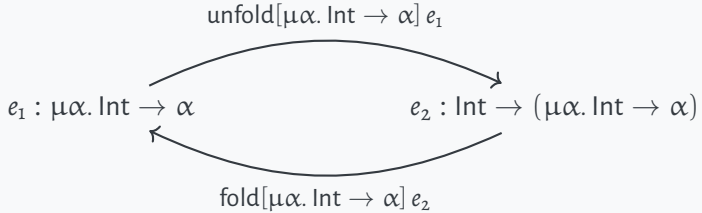

- unfold and fold operators are eliminated in the operational semantics.

$$\mathsf{unfold}[A](\mathsf{fold}[A]v) \hookrightarrow v$$

Iso-Recursive Subtyping

It is useful to have subtyping for iso-recursive types

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class ColorPoint {
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As Recursive Types

`ColorPoint` can be used wherever `Point` is expected.

$$\mu\alpha.\left\{\begin{array}{l}\text{color} : \text{String},\\ \text{x} : \text{Int},\\ \text{y} : \text{Int},\\ \text{move} : \text{Int} \to \text{Int} \to \alpha\end{array}\right\}$$

$$\leqslant \mu\alpha.\left\{\begin{array}{l}\text{x} : \text{Int},\\ \text{y} : \text{Int},\\ \text{move} : \text{Int} \to \text{Int} \to \alpha\end{array}\right\}$$

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Subtyping can be tricky

We need rules to determine when $\mu\alpha.\, A \leqslant \mu\alpha.\, B$.

Challenges in Iso-Recursive Subtyping: Examples

A desirable property: **Unfolding Lemma** for type soundness

$$\text{If } \mu\alpha.\, A \leqslant \mu\alpha.\, B, \text{ then } A[\mu\alpha.\, A/\alpha] \leqslant B[\mu\alpha.\, B/\alpha].$$

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- **Positive occurrences** of α: Usually straightforward.
  - $\mu\alpha.\, \top \to \alpha \leqslant \mu\alpha.\, \mathsf{Int} \to \alpha$ (✔, assumes $\mathsf{Int} \leqslant \top$ and recursive check)
  - Unfolds to: $\top \to (\mu\alpha.\, \top \to \alpha) \leqslant \mathsf{Int} \to (\mu\alpha.\, \mathsf{Int} \to \alpha)$

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- **Negative occurrences** of α: More complex.
  - $\mu\alpha.\, \alpha \to \mathsf{Int} \not\leqslant \mu\alpha.\, \alpha \to \top$ (✘)
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- **Negative occurrences with $\top$ / Reflexivity**:
  - $\mu\alpha.\, \alpha \to \alpha \leqslant \mu\alpha.\, \alpha \to \alpha$ (✔, by reflexivity)
  - $\mu\alpha.\, \top \to \alpha \leqslant \mu\alpha.\, \alpha \to \alpha$ (✔, if $\alpha \leqslant \top$ allowed in unfolding)
  - Unfolds to: $\top \to (\mu\alpha.\, \top \to \alpha) \leqslant (\mu\alpha.\, \alpha \to \alpha) \to (\mu\alpha.\, \alpha \to \alpha)$

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- **Nested Recursive Types**: $\mu\beta.\, \top \to (\mu\alpha.\, \alpha \to \beta)$? $\mu\beta.\, \mathsf{Int} \to (\mu\alpha.\, \alpha \to \beta)$ (✘)

Traditional Approach: The Amber Rules

The classical formulation for iso-recursive subtyping[1] [2].

S-Amber

$$\frac{\Gamma, \alpha \leqslant \beta \vdash A \leqslant B}{\Gamma \vdash \mu\alpha.\, A \leqslant \mu\beta.\, B}$$

S-Assmp

$$\frac{\alpha \leqslant \beta \in \Gamma}{\Gamma \vdash \alpha \leqslant \beta}$$

S-Refl

$$\frac{}{\Gamma \vdash \mu\alpha.\, A \leqslant \mu\alpha.\, A}$$

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$\Rightarrow$ an alternative, more tractable iso-recursive subtyping formulation.

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Alternative: Nominal Unfolding Rules

An approach that addresses Amber rules' issues[3].

- "Unfolding twice" triggers enough comparisons in covariant and contravariant positions.
- Simulates a "double unfolding" using *labeled types* A^{γ}.
- A fresh label γ is used for each recursive comparison.

S-Nominal

$$\frac{\Gamma, \alpha \vdash A[\alpha \mapsto A^{\gamma}] \leqslant B[\alpha \mapsto B^{\gamma}] \quad (\gamma \text{ fresh})}{\Gamma \vdash \mu\alpha.\, A \leqslant \mu\alpha.\, B}$$

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Benefits

Type sound, same expressive power as Amber rules, transitivity, simpler metatheory.
($\Rightarrow$ *Foundation for this work* $F^{\mu}_{\leqslant}$).

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Nominal Unfolding Rules: Example

S-Nominal (Reminder)

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- Covariant: $\Gamma, \alpha \vdash \text{nat} \leqslant \top$ (✔)
- Contravariant: $\Gamma, \alpha \vdash (\alpha \to \top)^{\gamma} \leqslant (\alpha \to \text{nat})^{\gamma}$
  - By S-Label: $\Gamma, \alpha \vdash \alpha \to \top \leqslant \alpha \to \text{nat}$
  - This requires $\Gamma, \alpha \vdash \top \leqslant \text{nat}$ (✘) and $\Gamma, \alpha \vdash \alpha \leqslant \alpha$ (✔).

Bounded Quantification: Quick Overview

Allows abstraction over types that satisfy a subtyping constraint (a bound).

- Example: $\forall(\alpha \leqslant \mathsf{Number}).\ \alpha \to \alpha$
- A function polymorphic in α, where α must be a subtype of Number.

[4] Pierce, "Bounded Quantification Is Undecidable".

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Kernel $F_{\leqslant}$

- $\forall(\alpha \leqslant A).\ B_1 \leqslant \forall(\alpha \leqslant A).\ B_2$
- Bounds must be identical (or equivalent).
- Decidable subtyping.

Full $F_{\leqslant}$

- $\forall(\alpha \leqslant A_1).\ B_1 \leqslant \forall(\alpha \leqslant A_2).\ B_2$
- Bounds can be contravariant ($A_2 \leqslant A_1$).
- More expressive, but undecidable[4].

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$F^{\mu}_{\leqslant}$ supports both variants: Kernel $F^{\mu}_{\leqslant}$ uses (equivalent bound) kernel $F_{\leqslant}$ subtyping rule; Full $F^{\mu}_{\leqslant}$ uses the full $F_{\leqslant}$ subtyping rule.

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Our Calculus: $F^{\mu}_{\leqslant}$

- $F^{\mu}_{\leqslant}$ integrates:
  - Iso-recursive subtyping (via Nominal Unfolding rules)
  - Bounded quantification (extending Kernel $F_{\leqslant}$ with equivalent bounds / Full $F_{\leqslant}$ with contravariant bounds)
- Types =
  - $F_{\leqslant}$ (bounded universal types $\forall(\alpha \leqslant A).\, B$) +
  - $\mu\alpha.\, A$ (recursive types) and A^{γ} (labeled types).
- Terms = $F_{\leqslant}$ terms + iso-recursive fold/unfolds

Types	$A, B, \ldots$	$::=$	$\mathsf{nat} \mid \top \mid A_1 \to A_2 \mid \alpha \mid \mu\alpha.\, A \mid A^{\gamma} \mid \forall(\alpha \leqslant A).\, B$
Expressions	e	$::=$	$x \mid \mathsf{i} \mid e_1\, e_2 \mid \lambda x : A.\, e \mid e\, A \mid \Lambda(\alpha \leqslant A).\, e \mid \mathsf{unfold}\ [A]\ e \mid \mathsf{fold}\ [A]\ e$
Values	v	$::=$	$\mathsf{i} \mid \lambda x : A.\, e \mid \Lambda(\alpha \leqslant A).\, e \mid \mathsf{fold}\ [A]\ v$

Our Calculus: $F^{\mu}_{\leqslant}$

- Subtyping rules: Combine $\mathbf{F}_{\leqslant}$ **rules** with **nominal unfolding** for μ-types.
  - The only change: recursive variable in $\mu\alpha.A$ is introduced bounded as $\alpha \leqslant \top$, so we have a unified type context.

$$\text{Contexts} \quad \Gamma \quad ::= \quad \cdot \mid \Gamma, \alpha \leqslant A \mid \Gamma, x : A$$

S-Forall(full)

$$\frac{\Gamma \vdash A_2 \leqslant A_1 \quad \Gamma, a \leqslant A_2 \vdash B_1 \leqslant B_2}{\Gamma \vdash \forall(a \leqslant A_1).\, B_1 \leqslant \forall(a \leqslant A_2).\, B_2}$$

S-Var-trans

$$\frac{\alpha \leqslant A \in \Gamma \quad \Gamma \vdash A \leqslant B}{\Gamma \vdash \alpha \leqslant B}$$

S-Nominal

$$\frac{\Gamma, \alpha \leqslant \top \vdash A[\alpha \mapsto A^{\gamma}] \leqslant B[\alpha \mapsto B^{\gamma}] \quad (\gamma \text{ fresh})}{\Gamma \vdash \mu\alpha.A \leqslant \mu\alpha.B}$$

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Metatheory: Main Properties of $F^{\mu}_{\leqslant}$

We prove key properties for both Kernel $F^{\mu}_{\leqslant}$ and Full $F^{\mu}_{\leqslant}$, formalized in Rocq:

- **Type soundness**
  - **Progress**: If $\Gamma \vdash e : A$, then e is either a value or exists a step $e \hookrightarrow e'$.
  - **Preservation**: If $\Gamma \vdash e : A$ and $e \hookrightarrow e'$, then $\Gamma \vdash e' : A$.
  - Key challenge: finding the correct generalization of the unfolding lemma $\mu\alpha.\, A \leqslant \mu\alpha.\, B \Rightarrow A[\mu\alpha.\, A/\alpha] \leqslant B[\mu\alpha.\, B/\alpha]$.

[5] Ghelli, 1993, "Recursive Types Are Not Conservative over F⩽".

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- **Conservativity** over $F_{\leqslant}$
  - If $\Gamma \vdash e : A$ holds in $F_{\leqslant}$, then $\Gamma \vdash e : A$ in $F^{\mu}_{\leqslant}$.
  - If $\Gamma \vdash e : A$ holds in $F^{\mu}_{\leqslant}$ and only contains $F_{\leqslant}$ constructs, then $\Gamma \vdash e : A$ also holds in $F_{\leqslant}$.
  - Not the case in equi-recursive types + $F_{\leqslant}$.[5]

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- **Conservativity** over $F_{\leqslant}$
  - If $\Gamma \vdash e : A$ holds in $F_{\leqslant}$, then $\Gamma \vdash e : A$ in $F^{\mu}_{\leqslant}$.
  - If $\Gamma \vdash e : A$ holds in $F^{\mu}_{\leqslant}$ and only contains $F_{\leqslant}$ constructs, then $\Gamma \vdash e : A$ also holds in $F_{\leqslant}$.
  - Not the case in equi-recursive types + $F_{\leqslant}$.[5]
- **Decidability** of typing & subtyping (for Kernel $F^{\mu}_{\leqslant}$)

[5] Ghelli, 1993, "Recursive Types Are Not Conservative over F⩽".

Comparison with Other Systems

Calculus	Recursive Types	Bounded Quant.	Decidability (Subtyping)	Transitivity (Subtyping)	Conservativity (over $F_{<:}$)	Modular Extension	Mechanized Proofs
Amadio et al. (1993)	Iso	N/A	✓	Built-in	N/A	×	×
Zhou et al. (2022)	Iso	N/A	✓	✓	N/A	✓	✓
Ghelli (1993)	Equi	Full		×	×	×	×
Colazzo et al. (2005)	Equi	Kernel	✓	✓		×	×
Jeffrey (2001)	Equi	Full	✓*	✓	×	×	×
Abadi et al. (1996)	Iso	Full		Built-in			×
Kernel $F^{\mu}_{\leqslant}$ (This work)	Iso	Kernel	✓	✓	✓	✓	✓
Full $F^{\mu}_{\leqslant}$ (This work)	Iso	Full	×	✓	✓	✓	✓

Table: Comparison among related works.

Application: Object Encodings

Objects can be encoded in typed lambda calculi in various ways[6].

- Recursive types: $\mu\alpha.\{\ldots \text{methods}(\alpha) \ldots\}$ **(OR)**

[6] Bruce, Cardelli, and Pierce, 1999, "Comparing Object Encodings".

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- Existential types: $\exists\beta.\beta \times (\beta \to \{\ldots \text{methods}(\beta) \ldots\})$ **(OE)**
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- Recursive + bounded existentials: $\mu\alpha.\exists(\beta \leqslant \alpha).\beta \times (\beta \to \{\ldots \text{methods}(\beta) \ldots\})$ **(ORBE)**

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$$\exists(\alpha \leqslant A).B \triangleq \forall(\beta \leqslant \top).\,(\forall(\alpha \leqslant A).\,B \to \beta) \to \beta$$

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Kernel $F^{\mu}_{\leqslant}$ encodes **OR**, **OE**, and **ORE**. Full $F^{\mu}_{\leqslant}$ encodes **ORBE**.

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Application: Precise Object Typing with Bounded Quantification

*Point Object Type in (**OR**)*

$$\text{Point} \triangleq \mu\alpha.\{\text{x} : \text{Int}, \text{y} : \text{Int}, \text{move} : \text{Int} \to \text{Int} \to \alpha\}$$

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function move1 [P <= Point] (p : P) = (unfold [Point] p).move 1 1
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F-bounded Polymorphism: The Challenge and Our Approach

- **F-bounded quantification** is a known technique to achieve self-type polymorphism[7], typically written as $\forall(\alpha \leqslant F[\alpha]).A$.
  - Example: $F_{\mathsf{Point}}(\alpha) \triangleq \{\mathsf{x} : \mathsf{Int}, \mathsf{y} : \mathsf{Int}, \mathsf{move} : \mathsf{Int} \to \mathsf{Int} \to \alpha\}$
  - move1 can have the type $\forall(\mathsf{P} \leqslant F_{\mathsf{Point}}(\mathsf{P})).\mathsf{P} \to \mathsf{P}$

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Key Design 1: Structural Folding/Unfolding Rules

Inspired by Abadi et al.[9], we use structural rules for typing 'fold/unfold' terms:

T-SUnfold (Structural Unfold)

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A \leqslant \mu\alpha.\, B}{\Gamma \vdash \mathsf{unfold}[A]e : B[\alpha \mapsto A]}$$

T-SFold (Structural Fold)

$$\frac{\Gamma \vdash e : A[\alpha \mapsto B] \quad \Gamma \vdash \mu\alpha.\, A \leqslant B}{\Gamma \vdash \mathsf{fold}[B]e : B}$$

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`(unfold [Point] p).move` has the type $\mathsf{Int} \to \mathsf{Int} \to \mathsf{P}$ instead of $\mathsf{Int} \to \mathsf{Int} \to \mathsf{Point}$ so `move1` has the type $\forall(\mathsf{P} \leqslant \mathsf{Point}).\, \mathsf{P} \to \mathsf{P}$.

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Application 2: Subtyping for Algebraic Data Types (ADTs)

The intentional behavior of ADTs can be encoded in $F^{\mu}_{\leqslant}$ using Scott encodings[10]:

```
data Exp1 = Num Int
          | Add Exp1 Exp1
          | Sub Exp1 Exp1
```

$$\mathsf{Exp}_1 \triangleq \mu\,\mathsf{E}.\,\forall(\mathsf{R} \leqslant \top).\ \left\{\begin{array}{l}\mathsf{num} : \mathsf{Int} \to \mathsf{R},\\ \mathsf{add} : \mathsf{E} \to \mathsf{E} \to \mathsf{R},\\ \mathsf{sub} : \mathsf{E} \to \mathsf{E} \to \mathsf{R}\end{array}\right\} \to \mathsf{R}$$

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Constructors (Num_1, Add_1) and case analysis functions (eval) are definable.

```
function Num₁ (n: Int) = fold [Exp₁] (Λ A. λ e. (e.num n))
function Add₁ (e₁ : Exp₁, e₂ : Exp₁) = fold [Exp₁] (Λ A. λ e. (e.add e₁ e₂))

function eval (e : Exp₁) = (unfold [Exp₁] e) [Int] {
  num = λn. n,
  add = λe₁ e₂. (eval e₁ + eval e₂),
  sub = λe₁ e₂. (eval e₁ - eval e₂)
}
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Similarly, we can define Exp_2 extending Exp_1 with a Neg constructor:

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Benefits of ADT Subtyping ($\mathsf{Exp}_1 \leqslant \mathsf{Exp}_2$)

- **Code Reuse**: A function processing Exp_2 ($\mathsf{eval}_2 : \mathsf{Exp}_2 \to \mathsf{Int}$) can operate on Exp_1 terms.
- **Polymorphic Constructors**: Avoid code duplication but preserve type precision.

ADTs: Polymorphic Constructors & Lower Bounds

Instead of writing individual constructors for each ADT:

```
function Add1 (e1 : Exp1, e2 : Exp1) = fold [Exp1] (Λ A. λ e. (e.add e1 e2))
function Add2 (e1 : Exp2, e2 : Exp2) = fold [Exp2] (Λ A. λ e. (e.add e1 e2))
```

ADTs: Polymorphic Constructors & Lower Bounds

Instead of writing individual constructors for each ADT:

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function Add₁ (e₁ : Exp₁, e₂ : Exp₁) = fold [Exp₁] (Λ A. λ e. (e.add e₁ e₂))
function Add₂ (e₁ : Exp₂, e₂ : Exp₂) = fold [Exp₂] (Λ A. λ e. (e.add e₁ e₂))
```

We can define a single constructor for both Exp_1 and Exp_2:

```
function Add∀ [E ⩾ Exp₁] (e₁ : E, e₂ : E) = fold [E] (Λ A. λ e. (e.add e₁ e₂))
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Using $\mathsf{Add}_{\forall} : \forall(\mathsf{E} \geqslant \mathsf{Exp}_1).\, \mathsf{E} \to \mathsf{E} \to \mathsf{E}$, we can construct terms of type Exp_1, Exp_2, etc., if they are all supertypes of some base expression type.

Required $F^{\mu}_{\leqslant}$ ingredients:

- **Structural folding rules** for fold expressions
- **Lower bounded quantification**, also studied in this work.

Key Design: Lower Bounded Quantification for ADTs

- Extends Kernel $F^{\mu}_{\leqslant}$ with:
  - Lower bounded quantification: $\forall(\alpha \geqslant A).\, B$ (type α is a supertype of A).
  - Note, a variable can only be either upper or lower bounded, not both.
  - Bottom type ($\bot$), single field record types ($\{\mathsf{l} : A\}$), and intersection types for records (&).

$$\{x : \mathsf{Int}, y : \mathsf{Int}\} \triangleq \{x : \mathsf{Int}\} \,\&\, \{y : \mathsf{Int}\}$$

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$$\{x : \mathsf{Int}, y : \mathsf{Int}\} \triangleq \{x : \mathsf{Int}\} \,\&\, \{y : \mathsf{Int}\}$$

- $F^{\mu\wedge}_{\leqslant\geqslant}$ successfully types polymorphic ADT constructors like $\mathsf{Add}_{\forall}$.
- Same metatheory results (type soundness, conservativity, decidability) proved for $F^{\mu\wedge}_{\leqslant\geqslant}$.

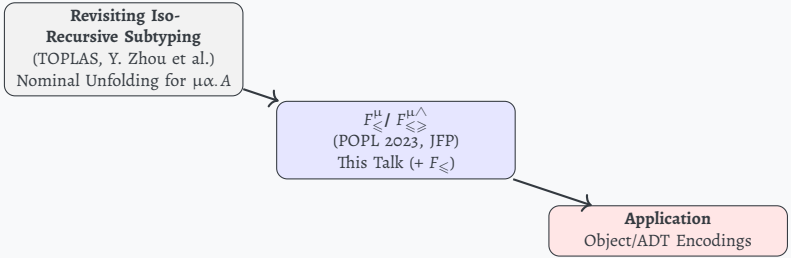
Conclusion

- Successfully integrated iso-recursive types (nominal unfolding) with kernel and full $F_{\leqslant}$.
  - **Kernel** $F^{\mu}_{\leqslant}$ **/** $F^{\mu\wedge}_{\leqslant\geqslant}$: Transitive, decidable subtyping, conservative over kernel $F_{\leqslant}$, type sound.
  - **Full** $F^{\mu}_{\leqslant}$: Transitive, undecidable (as expected) subtyping, conservative over full $F_{\leqslant}$, type sound.
- Applications:
  - Foundational calculus for well-known object encodings
  - Encoding positive forms of F-bounded quantification
  - Enable subtyping for algebraic data types (ADTs) and polymorphic ADT constructors

Other work on iso-recursive types

Revisiting Iso-Recursive Subtyping
(TOPLAS, Y. Zhou et al.)
Nominal Unfolding for $\mu\alpha.A$

Other work on iso-recursive types



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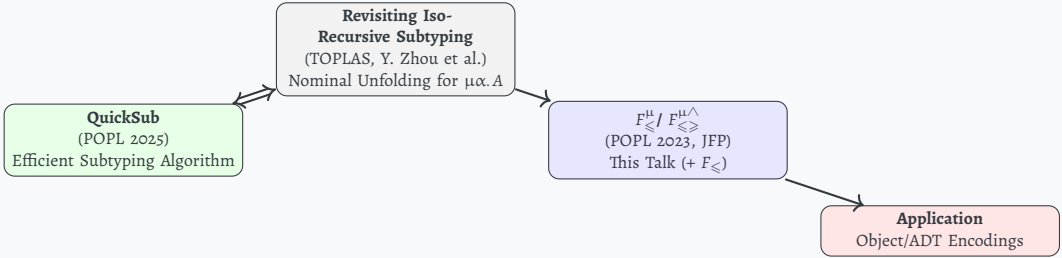

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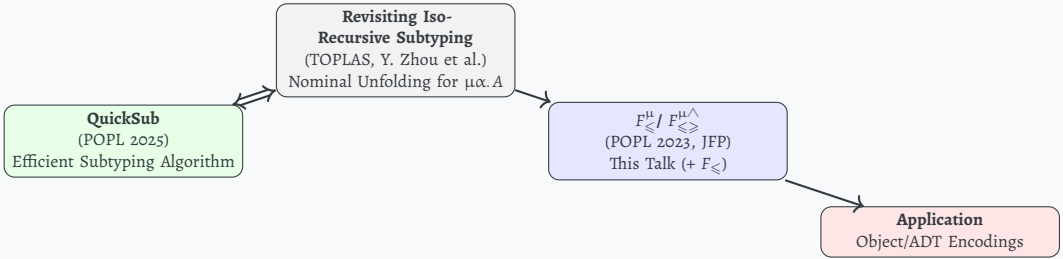
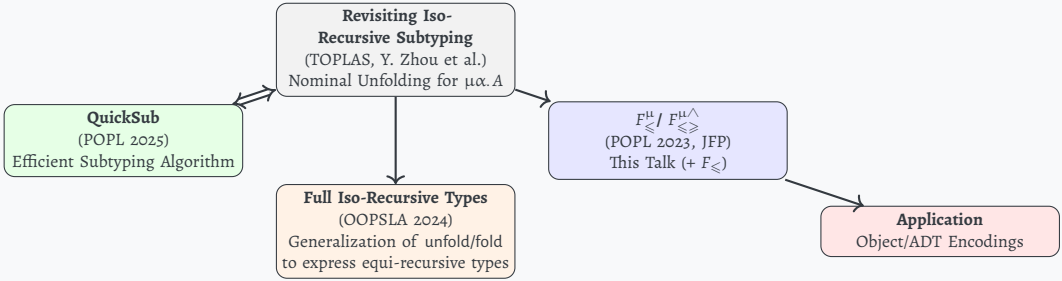


QuickSub: Efficient Iso-Recursive Subtyping

$$\Psi \vdash_{\oplus} A \lessapprox B$$

- Ideas: Tracking polarity $\oplus$, distinguish strict subtypes with equivalence on the fly ($\lessapprox ::= < \mid \approx_S$).
- Equivalent to Amber rules, linear complexity for practical subtypes

Other work on iso-recursive types



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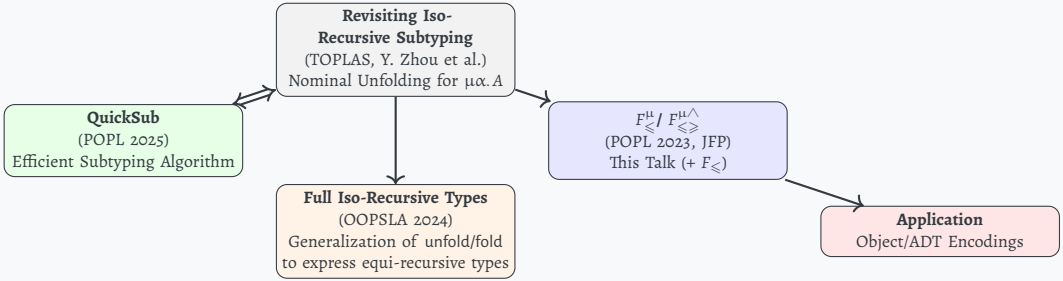
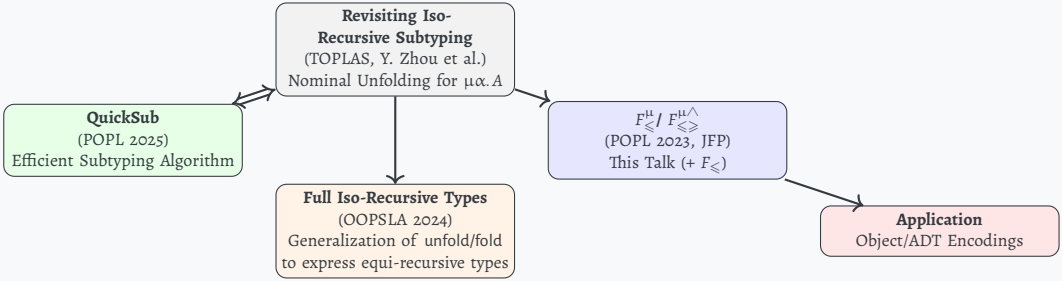


Full Iso-Recursive Types

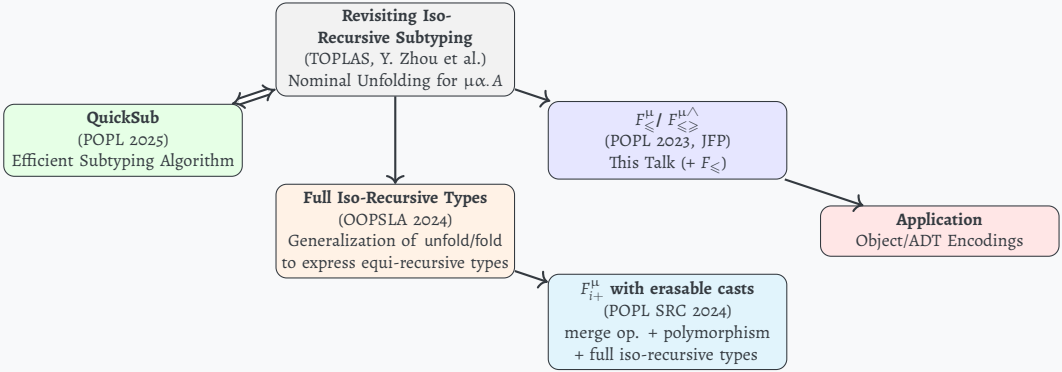
$$e ::= \ldots \mid \mathsf{cast}[c]e \qquad c ::= \mathsf{unfold}_{\mu\alpha.A} \mid \mathsf{fold}_{\mu\alpha.A} \mid c_1 \to c_2 \mid \mathsf{fix}\ \iota.c \mid \ldots$$

- Idea: Replace the standard unfold/fold operator with cast's, to enable deep isomorphic transformations on recursive types.
- Same expressiveness of equi-recursive types (easy proof!), casts are runtime irrelevant

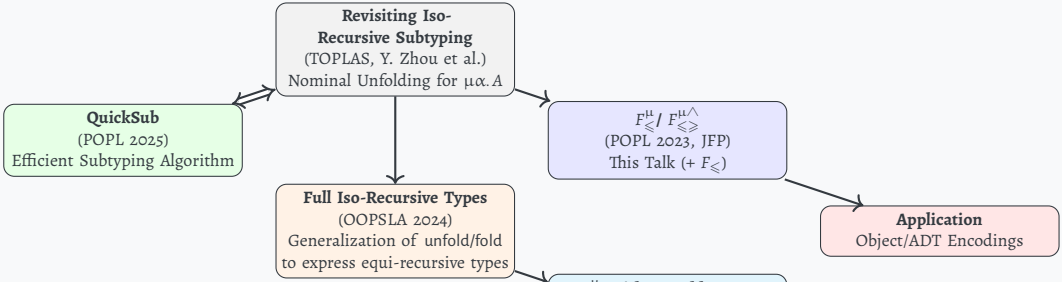
Other work on iso-recursive types



Other work on iso-recursive types



Other work on iso-recursive types

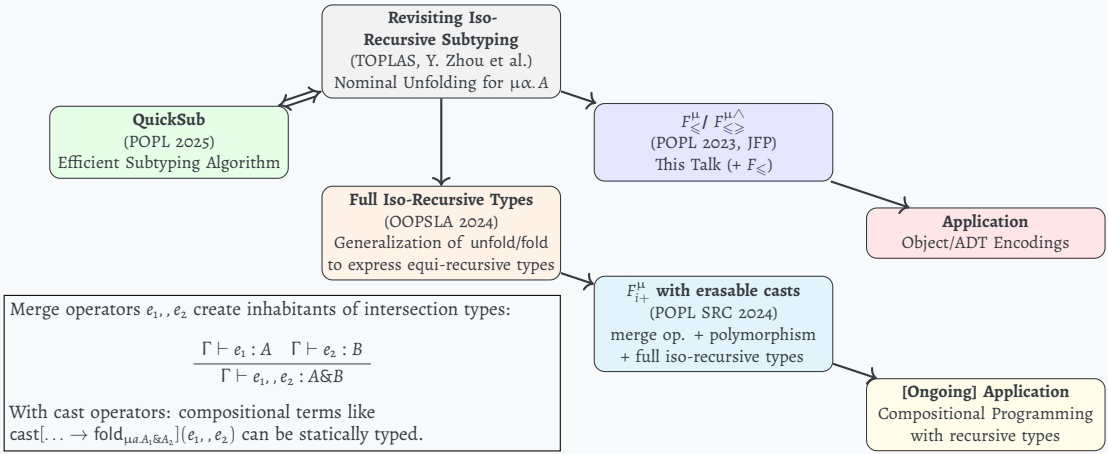


Merge operators $e_1, , e_2$ create inhabitants of intersection types:

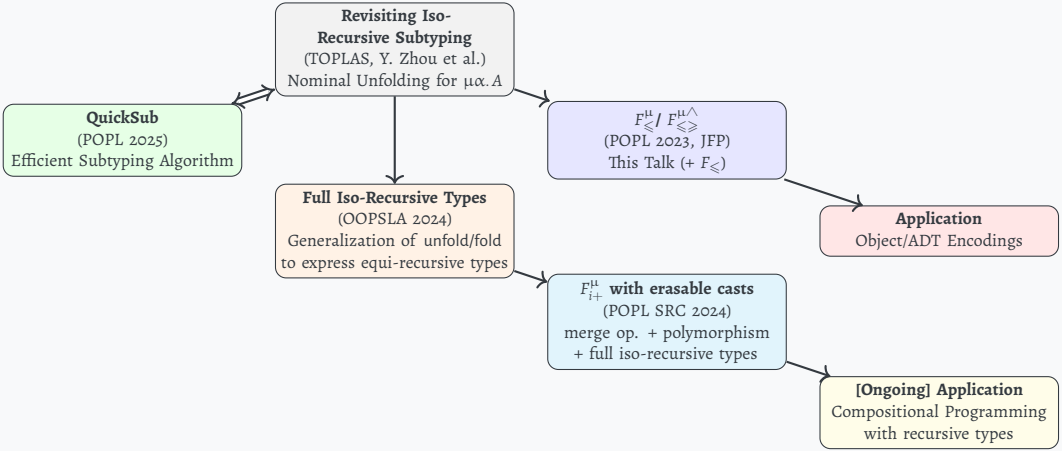
$$\frac{\Gamma \vdash e_1 : A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1, , e_2 : A \& B}$$

With cast operators: compositional terms like
$\text{cast}[\ldots \to \text{fold}_{\mu\alpha.A_1 \& A_2}](e_1, , e_2)$ can be statically typed.

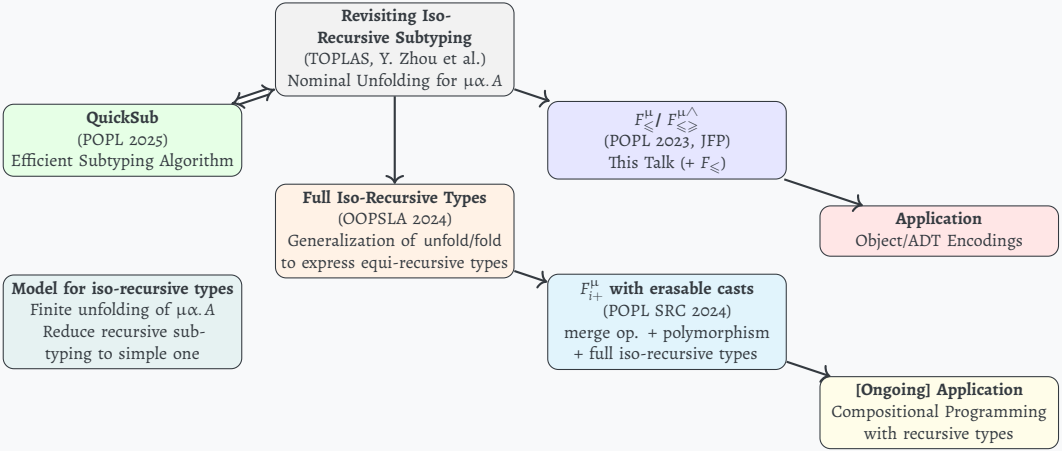
Other work on iso-recursive types



Other work on iso-recursive types

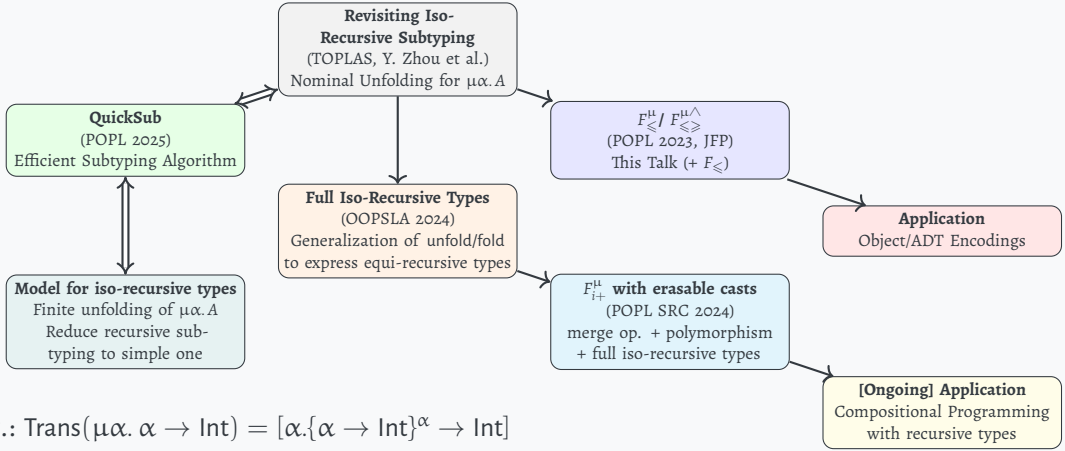


Other work on iso-recursive types



Idea: First translate recursive types into their (nominal) unfolding and then use the translation as a model for subtyping. $A \leqslant B \triangleq \mathsf{Trans}(A) <: \mathsf{Trans}(B)$.

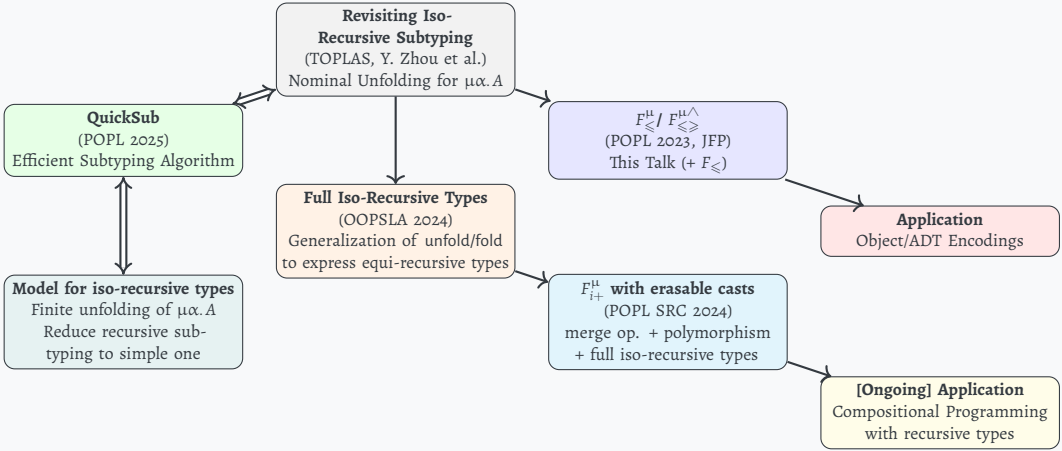
Other work on iso-recursive types



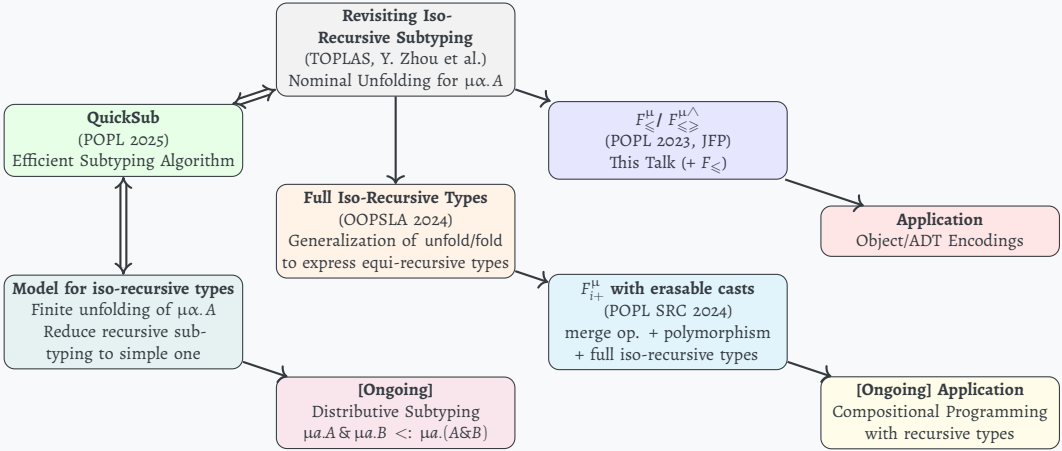
e.g.: $\mathsf{Trans}(\mu\alpha.\ \alpha \to \mathsf{Int}) = [\alpha.\{\alpha \to \mathsf{Int}\}^{\alpha} \to \mathsf{Int}]$

Idea: First translate recursive types into their (nominal) unfolding and then use the translation as a model for subtyping. $A \leqslant B \triangleq \mathsf{Trans}(A) <: \mathsf{Trans}(B)$.

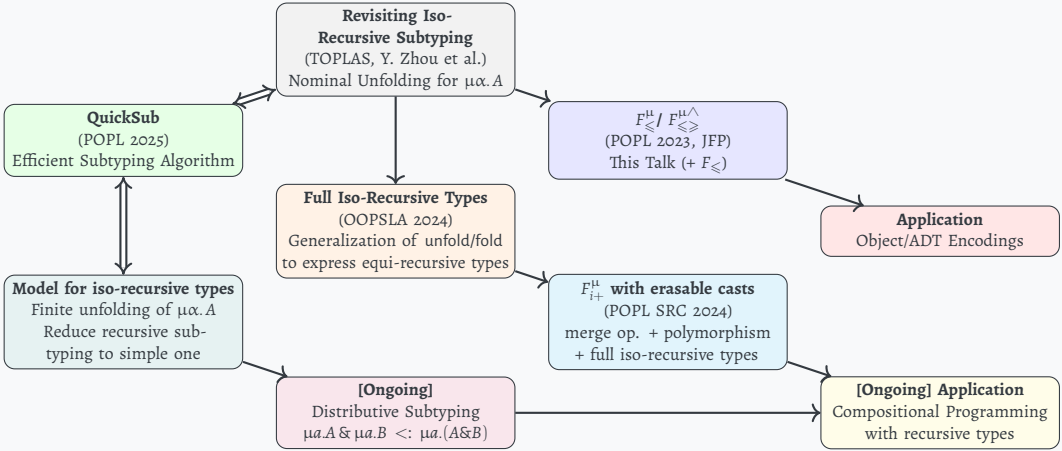
Other work on iso-recursive types



Other work on iso-recursive types



Thank you!


Revisiting Iso-Recursive Subtyping
(TOPLAS, Y. Zhou et al.)
Nominal Unfolding for $\mu\alpha.A$
QuickSub
(POPL 2025)
Efficient Subtyping Algorithm
$F^{\mu}_{\leqslant}$ / $F^{\mu\wedge}_{\leqslant\geqslant}$
(POPL 2023, JFP)
This Talk (+ $F_{\leqslant}$)
Full Iso-Recursive Types
(OOPSLA 2024)
Generalization of unfold/fold to express equi-recursive types
Application
Object/ADT Encodings
Model for iso-recursive types
Finite unfolding of $\mu\alpha.A$
Reduce recursive subtyping to simple one
F^{μ}_{i+} **with erasable casts**
(POPL SRC 2024)
merge op. + polymorphism
+ full iso-recursive types
[Ongoing]
Distributive Subtyping
$\mu a.A \,\&\, \mu a.B <: \mu a.(A\&B)$
[Ongoing] Application
Compositional Programming with recursive types

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