QuickSub: Efficient Iso-Recursive Subtyping

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Litao Zhou, Bruno C. d. S. Oliveira

University of Hong Kong

Recursive Types

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Types $A ::= \mathrm{Int} \mid \top \mid A_1 \to A_2 \mid \mu\alpha.A \mid \alpha \mid \ldots$

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class A {
  foo(x: Int) : A
  bar(x: A) : Int
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}
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Two main approaches: iso-recursive and equi-recursive types.

Iso-Recursive Types

Unlike equi-recursive types, recursive types and their unfoldings are not equal, and require explicit fold and unfold operations.

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$\mathsf{unfold}\,[\mu\alpha.\,\mathsf{Int} \to \alpha]\; e_1$

$e_1 : \mu\alpha.\,\mathsf{Int} \to \alpha$ $\qquad$ $e_2 : \mathsf{Int} \to (\mu\alpha.\,\mathsf{Int} \to \alpha)$

$\mathsf{fold}\,[\mu\alpha.\,\mathsf{Int} \to \alpha]\; e_2$

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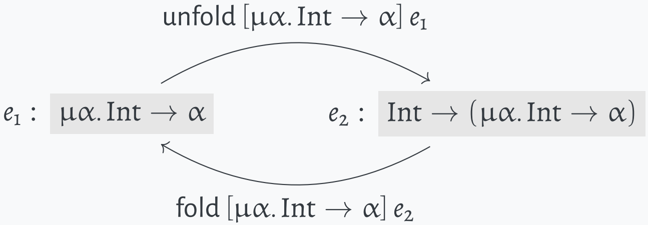

Iso-Recursive Subtyping

✓ $\mu\alpha.\,\top \to \alpha \leqslant \mu\alpha.\,\mathrm{Int} \to \alpha$

Recursive Subtyping

Iso-Recursive Subtyping (vs. Equi-Recursive Subtyping)

- ✓ Simpler metatheory
  - No coinductive reasoning is needed
- ✓ Easier to scale to more features[a,b,c]
- ✓ More efficient meta operations[d] (e.g. equivalence checking)

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Considerations for subtyping iso-recursive types

Unfolding Lemma (expected)

If $\mu\alpha. A \leqslant \mu\alpha. B$, then $A[\mu\alpha. A/\alpha] \leqslant B[\mu\alpha. B/\alpha]$.

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    - $(\mu\alpha. \alpha \to \mathsf{Int}) \to \boxed{\mathsf{Int}} \leqslant (\mu\alpha. \alpha \to \top) \to \boxed{\top}$ ✓
    - $((\mu\alpha. \alpha \to \mathsf{Int}) \to \boxed{\mathsf{Int}}) \to \mathsf{Int} \leqslant ((\mu\alpha. \alpha \to \top) \to \boxed{\top}) \to \top$ ✗

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  - What is the subtyping relation between $\mu\beta. \top \to (\mu\alpha. \alpha \to \beta)$ and $\mu\beta. \mathrm{Int} \to (\mu\alpha. \alpha \to \beta)$?

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Why iso-recurisve subtyping have been inefficient?

Amber Rules[1,2]

(Amber-rec)
$$\frac{\Gamma, \alpha \leqslant \beta \vdash A \leqslant B}{\Gamma \vdash \mu\alpha. A \leqslant \mu\beta. B}$$

(Amber-assump)
$$\frac{\alpha \leqslant \beta \in \Gamma}{\Gamma \vdash \alpha \leqslant \beta}$$

(Amber-self)
$$\frac{}{\Gamma \vdash \mu\alpha. A \leqslant \mu\alpha. A}$$

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  - $\mu\alpha. \alpha \to \mathrm{Int} \leqslant \mu\alpha. \alpha \to \mathrm{Int}$
- Amber-rec requires variable names to be distinct, which is non-trivial and requires extra runtime overhead for renaming.

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QuickSub: Efficient Iso-Recursive Subtyping

$$\Psi \vdash_{\oplus} A \lesssim B$$

Though written as a judgment, QuickSub can be easily interpreted as an algorithm.

- Input: subtyping context Ψ, polarity mode $\oplus$, types A and B.
- Output: subtyping result ($\lesssim ::= < \mid \approx_S$), or failure where no rules apply.
- Equivalent to Amber rules.

Key idea (1) - tracking polarity

Subtyping Context	Ψ	$::= \Psi, \alpha^{\oplus} \mid \cdot$
Polarity Mode	$\oplus$	$::= + \mid -$
Subtyping Results	$\lessapprox$	$::= < \mid \approx$

$$\boxed{\Psi \vdash_{\oplus} A \lessapprox B}$$

(QSub-RecLt)

$$\frac{\Psi, \alpha^{\oplus} \vdash_{\oplus} A < B}{\Psi \vdash_{\oplus} \mu\alpha. A < \mu\alpha. B}$$

(QSub-Fun)

$$\frac{\Psi \vdash_{\overline{\oplus}} A_2 \lessapprox_1 B_2 \quad \Psi \vdash_{\oplus} A_1 \lessapprox_2 B_1}{\Psi \vdash_{\oplus} A_1 \to A_2 (\lessapprox_1 \bullet \lessapprox_2) B_1 \to B_2}$$

When $\alpha^{\oplus}$ is the same as $\vdash_{\oplus}$, α is positive (to the right of $\to$'s)
When $\alpha^{\overline{\oplus}}$ is the flip of $\vdash_{\oplus}$, α is negative (to the left of $\to$'s)

Key idea (2) - distinguish strict subtypes from equivalence

Subtyping results are precisely tracked from the base cases:

(QSub-Int)
$$\frac{}{\Psi \vdash_{\oplus} \mathsf{Int} \approx \mathsf{Int}}$$

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The results are composed accordingly in the function case:

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Positive variables can be simply considered equal.

(QSub-VarPos)
$$\frac{\alpha^{\oplus} \in \Psi}{\Psi \vdash_{\oplus} \alpha \approx \alpha}$$

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Observation: When negative recursive variables are compared with themselves, their recursive types have to be equivalent.

causes failure

$$\dfrac{\dfrac{\alpha^{+} \vdash_{-} \alpha \approx \alpha \qquad \alpha^{+} \vdash_{+} \mathsf{Int} < \top}{\alpha^{+} \vdash_{+} \alpha \to \mathsf{Int} < \alpha \to \top}}{\cdot \vdash_{+} \mu\alpha.\alpha \to \mathsf{Int} \not< \mu\alpha.\alpha \to \top} \; ✗$$

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Key idea: equality variable set

Subtyping Context	Ψ	$::= \Psi, \alpha^{\oplus} \mid \cdot$
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Subtyping Results	$\lessapprox$	$::= < \mid \approx_S$
Equality Var. Set	S	$::= \emptyset \mid \{\alpha_1, \ldots, \alpha_n\}$

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$$\begin{aligned} \approx_{S_1} \bullet \approx_{S_2} &= \approx_{S_1 \cup S_2} \\ \approx_{\emptyset} \bullet < &= < \\ < \bullet < &= < \\ < \bullet \approx_{\emptyset} &= < \end{aligned}$$

Otherwise, $\lessapprox_1 \bullet \lessapprox_2$ fails

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(QSub-RecEq)

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✓ $\mu\alpha. \alpha \to \text{Int} \nleq \mu\alpha. \alpha \to \top$

✓ Reflexive subtyping.

QuickSub in Functional Style

$$
\begin{array}{lcll}
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\mathsf{Sub}_{\Psi}(\top, \top, \oplus) & = & \approx_{\emptyset} & \\
\mathsf{Sub}_{\Psi}(A, \top, \oplus) & = & < & (\text{if } A \neq \top) \\
\mathsf{Sub}_{\Psi}(\alpha, \alpha, \oplus) & = & \approx_{\emptyset} & (\text{if } \alpha^{\oplus} \in \Psi) \\
\mathsf{Sub}_{\Psi}(\alpha, \alpha, \oplus) & = & \approx_{\{\alpha\}} & (\text{if } \alpha^{\overline{\oplus}} \in \Psi) \\
\mathsf{Sub}_{\Psi}(A_1 \to A_2, B_1 \to B_2, \oplus) & = & \mathsf{Sub}_{\Psi}(A_2, A_1, \overline{\oplus}) \bullet \mathsf{Sub}_{\Psi}(B_1, B_2, \oplus) & \\
\mathsf{Sub}_{\Psi}(\mu\alpha.\, A_1, \mu\alpha.\, A_2, \oplus) & = & < & (\text{if } \mathsf{Sub}_{\Psi, \alpha^{\oplus}}(A_1, A_2, \oplus) = <) \\
\mathsf{Sub}_{\Psi}(\mu\alpha.\, A_1, \mu\alpha.\, A_2, \oplus) & = & \approx_{S} & (\text{if } \mathsf{Sub}_{\Psi, \alpha^{\oplus}}(A_1, A_2, \oplus) = \approx_{S} \text{ and } \alpha \notin S) \\
\mathsf{Sub}_{\Psi}(\mu\alpha.\, A_1, \mu\alpha.\, A_2, \oplus) & = & \approx_{(S \cup \mathsf{FV}(A_1)) \setminus \{\alpha\}} & (\text{otherwise})
\end{array}
$$

otherwise, $\mathsf{Sub}_{\Psi}(A, B, \oplus)$ fails

- No backtracking $\Rightarrow O(n)$ traversal ($n =$ size of types)
- Set operations can be optimized with imperative data structures
  - $\Rightarrow O(m)$ cost in the worst case ($m = \#$ of recursive variables)
  - $\Rightarrow O(1)$ for practical cases
  - $\Rightarrow$ Overall: $O(mn)$ cost in the worst case, linear for practical cases

Evaluation

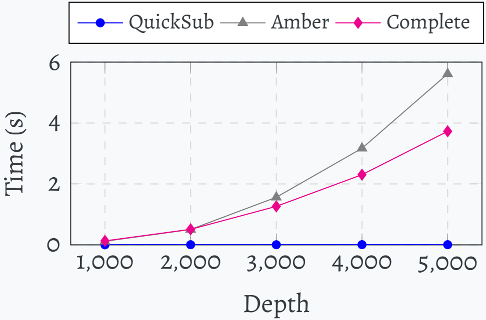
- Implement QuickSub in OCaml.
- Compare performance with existing algorithms:
  - Amber rules
  - Nominal unfolding rules[1]
    - Equivalent to Amber rules, addressing metatheory challenges.
    - Though being algorithmic, not designed with efficiency in mind.
  - Complete iso-recursive subtyping[2].
    - More expressive than Amber rules.
    - Ship with an $O(mn)$ algorithm.
  - Equi-recursive subtyping[3] (see paper).
- Benchmarks for different recursive type patterns and depths.

[1] Y. Zhou et al., "Revisiting Iso-recursive subtyping".
[2] Ligatti et al., "On subtyping-relation completeness, with an application to iso-recursive types".
[3] Gapeyev et al., "Recursive subtyping revealed".

Nested positive subtyping, growing depths

```
class A {
  foo(x: Int) : A
   ...
   class B {
     bar(y: Real) : A
   }
}
```

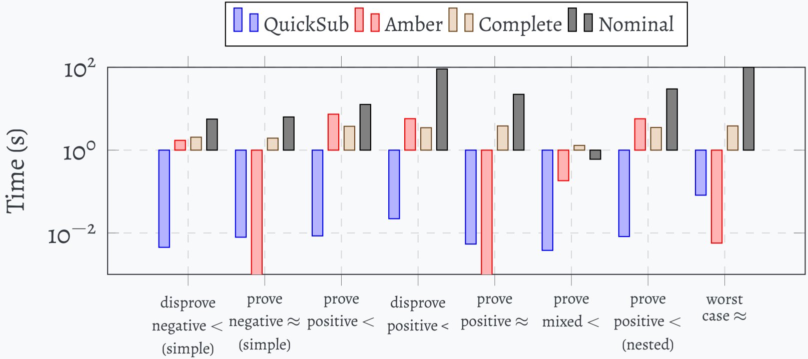
Complex Nested Objects



$$\mu\alpha_1.\,\mathrm{Int} \to (\mu\alpha_2.\,\mathrm{Int} \to \ldots(\mu\alpha_n.\,(\alpha_1, \ldots, \alpha_n)))$$

For positive nested recursive subtyping, Amber and Complete are quadratic in complexity, while QuickSub is linear.

Benchmark results, algorithm runtime at a large depth

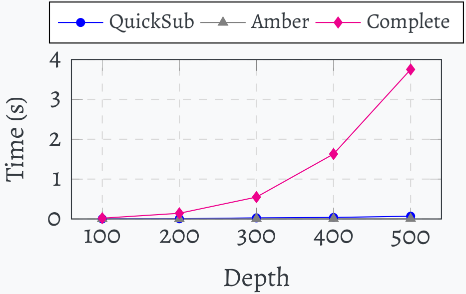


- QuickSub outperforms other algorithms in most cases
  - except in reflexive cases, where Amber performs faster (expected)
- Handles both simple and nested recursive types efficiently.
- Linear performance in practical scenarios.

Negative recursive subtyping, worst case

$$\mu\alpha_1.\ \alpha_1 \to (\mu\alpha_2.\ \alpha_1 \to \alpha_2 \to (\mu\alpha_3.\ \alpha_1 \to \alpha_2 \to \alpha_3 \to (\mu\alpha_4.\ \ldots \ldots)))$$

Worst Case Pattern



The worst case scenario only occurs when all variables are negative and the subtyping result is $\approx$, so that all variables are added to the equality variable set S. ($|S|_{\max} = m$)

The complexity is $O(mn)$. ($m = \#$ of variables, $n =$ size of types).

With the imperative optimization, QuickSub still demonstrates an efficient performance.

Conclusion

QuickSub, an efficient algorithm for iso-recursive subtyping, with linear complexity in practice.

More in the paper

- Equivalence proof to other iso-recursive subtyping formulations.
- Type soundness proof for a calculus using QuickSub.
- Extension to record types.

Future Work

- Extending QuickSub to handle more type system features.
- Applying QuickSub to deal with equi-recursive subtyping.

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